# Group cohomology

Today:

- Short motivation
- definitions
- How to compute + low dim. cohomology groups
- computational tools, GAP, ..
- Hochschild cohomology

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Motivation:

- Group cohomology (of $G$) encodes information about $G$ and the modules it acts on.
- Group actions appear everywhere in math, group cohomology becomes a powerful tool to extract information in such situations. For example in alg. top. and number theory, many theories depend heavily on group cohomology, for example class field theory.

Definition:

- $G$ a (discrete) group,
- $A$ a $G$-module (abelian group with $G$-action compatible with gp str.) $= \mathbb{Z}G$-module.

Then we have functors

$$G\text{-mod} \longrightarrow Ab$$

$$(-)^G \qquad A \longmapsto A^G = \{a \in A : ga = a \ \forall g \in G\},$$

$$(-)_G \qquad A \longmapsto A_G := A / \langle ga - a : g \in G, a \in A \rangle.$$

Note that $(-)^G = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$, which is left exact and $(-)_G = (-) \otimes_{\mathbb{Z}G} \mathbb{Z}$, which is right exact. Hence we have a right derived functor $R\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$ and a left derived functor $L(- \otimes_{\mathbb{Z}G} \mathbb{Z})$ and we define

**Def:**

$$H^i(G, A) := H^i(R\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A)$$

$$H_i(G, A) := H_i(L(A \otimes_{\mathbb{Z}G} \mathbb{Z})) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A)$$

We will see that both functors can be computed via the same resolution!

## How to compute them?

First we recall that the groups $\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A)$ can be computed by taking a projective resolution of $\mathbb{Z}$ (e.g. a free resolution).

Then we note that $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A)$ can be computed either via an injective resolution of $A$ or as a projective resolution of $\mathbb{Z}$:

Indeed, the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ may be viewed as a covariant functor

$$(\mathrm{Mod}_{\mathbb{Z}G})^{op} \longrightarrow \mathrm{Ab}$$

which is again left exact and we consider $R\mathrm{Hom}_{\mathbb{Z}G}(-, A)$. Since $\mathrm{Mod}_{\mathbb{Z}G}$ has enough projectives (and injectives) we get

$$R\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A) = \mathrm{Ext}^i_{\mathrm{Mod}_{\mathbb{Z}G}}(\mathbb{Z}, A) := \mathrm{Hom}_{D(\mathrm{Mod}_{\mathbb{Z}G})}(\mathbb{Z}[0], A[i])$$

which can be computed by taking a projective resolution $P \longrightarrow \mathbb{Z}$:

$$\mathrm{Hom}_{D(\mathrm{Mod}_{\mathbb{Z}G})}(\mathbb{Z}[0], A[i]) \simeq \mathrm{Hom}_{D(\mathrm{Mod}_{\mathbb{Z}G})}(P, A[i])$$

$$\simeq \mathrm{Hom}_{K(\mathrm{Mod}_{\mathbb{Z}G})}(P, A[i]) \simeq H^i(\mathcal{H}om^\bullet(P, A))$$

(Morph. of compl. modulo homotopy equiv.)

$$= H^i(\mathrm{Hom}_{\mathbb{Z}G}(P_0, A) \to \mathrm{Hom}_{\mathbb{Z}G}(P_{-1}, A) \to \cdots)$$

What is good about this approach is that we can use <u>one</u> projective resolution to compute both $H_i(G, A)$ and $H^i(G, A)$.

## The bar complex:

The following is a free (hence projective) resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module: ($\varepsilon(g)=1 \ \forall g \in G$)

$$\mathbb{Z}G^{\otimes\bullet} := ( \dots \to \overset{2}{\bigoplus_{G^2}} \mathbb{Z}G \xrightarrow{d} \overset{1}{\bigoplus_{G^1}} \mathbb{Z}G \xrightarrow{d} \overset{0}{\bigoplus_{G^0}} \mathbb{Z}G = \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

with $d = \sum_{i=0}^{n} (-1)^i d_i$ and

$$d_i(g_1, \dots, g_n) = \begin{cases} g_1(g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 1 \leq i \leq n-1 \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}.$$

## Low dimensional (co)homology groups:

$$\mathbb{Z}G^{\otimes\bullet} \otimes_{\mathbb{Z}G} A \cong ( \dots \xrightarrow{d} \bigoplus_{G^2} A \xrightarrow{d} \bigoplus_{G} A \xrightarrow{d} A )$$

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G^{\otimes\bullet}, A) = ( \underset{\underset{\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)}{\cong}}{A} \xrightarrow{d^\vee} \bigoplus_{G} A \xrightarrow{d^\vee} \bigoplus_{G^2} A \xrightarrow{d^\vee} \dots )$$

$\rightsquigarrow$ view as map $\mathbb{Z}G \to A$

**dim 0:** $d(g \otimes a) = ga - a$

$$d^\vee(a)(g) = a \circ (d_0 - d_1)(g) = a(g-1) = ga - a$$

$$\Rightarrow \boxed{\begin{aligned} H^0(G, A) &= \{ a \in A : ga - a = 0 \} \\ H_0(G, A) &= A / \langle ga - a : g \in G, a \in A \rangle \end{aligned}}$$

**dim 1:** $d(g_1 \otimes g_2 \otimes a) = g_2 \otimes g_1 a - g_1 g_2 \otimes a + g_1 \otimes a$

$$(d^\vee \varphi)(g_1, g_2) = \varphi \circ (d_0 - d_1 + d_2)(g_1, g_2)$$

$$= \varphi \circ (g_1(g_2) - (g_1 g_2) + (g_2))$$

$$= g_1 \varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1)$$

$$\Longrightarrow \boxed{\begin{array}{l} H^1(G,A) = \dfrac{\{\varphi : G \to A \mid \varphi(g_1 g_2) = \varphi(g_1) + g_1 \varphi(g_2)\}}{\{p_a : G \to A \mid g \mapsto ga - a \text{ for } a \in A\}} \\ \\ H_1(G,A) \quad \text{complicated.} \end{array}}$$

dim 2: $(d^\vee \varphi)(g_1, g_2, g_3) = \varphi \circ (d_0 - d_1 + d_2 - d_3)(g_1, g_2, g_3)$

$$= g_1 \varphi(g_2, g_3) - \varphi(g_1 g_2, g_3) + \varphi(g_1, g_2 g_3) - \varphi(g_1, g_2)$$

$$\Longrightarrow \boxed{H^2(G,A) = \{\varphi : G^2 \to A \mid \varphi(g_1, g_2) + \varphi(g_1 g_2, g_3) = g_1 \varphi(g_2, g_3) + \varphi(g_1, g_2 g_3)\}}$$

Encodes associativity.

$H^2(G,A)$ classifies extensions $0 \to A \to E \to G \to 1$:

Given a 2-cocycle $\varphi : G^2 \to A$ we may define a group $A \times_\varphi G$ with underlying set $A \times G$ and group structure given by

$$\boxed{(x, g) + (x', g') = (x + g x' + \varphi(g, g'), g g')}$$

The cocycle condition exactly gives us that this addition is associative.

Two extensions are equivalent if there is an iso. $E \xrightarrow{\sim} E'$ s.t. the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E & \to & G & \to & 1 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & A & \to & E' & \to & G & \to & 1. \end{array}$$

There is a group structure on the set of such extensions ($\mathrm{Ext}(G,A)/\sim$) given by (Baer sum)

$$E + E' = E \times_G E' / \{(-a, a) : a \in A\}.$$

**Theorem:** The map

$$H^2(G,A) \longrightarrow \mathrm{EXT}(G,A)/\sim$$

$$\varphi \longmapsto A \times_\varphi G$$

is an isomorphism of abelian groups.

**Rmk:** Group cohomology is a special case of Hochschild cohomolog (see notes on Hochschild cohomology).

# Hochschild Cohomology

Let $R$ be an algebra over a commutative ring $k$ and let $M$ be an $R$-$R$-bimodule, or equivalently, $M$ is a left module for the algebra $R^e := R \otimes R^{op}$ acting via $(a \otimes b).m := amb$. The Hochschild homology and cohomology are defined respectively by

$$\boxed{\begin{aligned} HH_*(R, M) &:= \mathrm{Tor}_*^{R^e/k}(M, R) \\ HH^*(R, M) &:= \mathrm{Ext}^*_{R^e/k}(R, M). \end{aligned}}$$ and

This generalizes group cohomology as follows: If we take $R = \mathbb{Z}G$ and declare $G$ to act trivially on $M$ as a right module then

as a bimodule

$$\mathrm{Tor}_*^{\mathbb{Z}G}(M, \mathbb{Z}) \cong \mathrm{Tor}_*^{\mathbb{Z}G^e}(M, \mathbb{Z}G)$$

$$\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M) \cong \mathrm{Ext}^*_{\mathbb{Z}G^e}(\mathbb{Z}G, M).$$

The Hochschild (co)homology is computed in a very similar fashion as we did for group (co)homology: we resolve $R$ as $R^{\otimes \bullet} \longrightarrow R$:

$$\cdots \longrightarrow \overset{1}{R^{\otimes 3}} \longrightarrow \overset{0}{R^{\otimes 2}} \longrightarrow \overset{-1}{R}$$

with differentials $d = \sum_{i=0}^{n} (-1)^i d_i$ where

$$d_i(r_0 \otimes \cdots \otimes r_n) = \begin{cases} r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & \text{if } 0 \le i \le n-1, \\ r_0 \otimes \cdots \otimes r_{n-1} & \text{if } i = n. \end{cases}$$

We claim that this complex computes the Hochschild (co)homology after applying $(\mathrm{Hom}_k(-,M))$ $M \otimes_k (-)$ [Wiebel, Lemma 9.1.3].

## GAP computations (see notebook)

We used gap to compute

$$\boxed{\begin{array}{ll} \dim_{\mathbb{F}_2} H^1(D_8, \mathbb{Z}/2\mathbb{Z}) = 2, & \dim_{\mathbb{F}_2} H^2(D_8, \mathbb{Z}/2\mathbb{Z}) = 3 \\ \dim_{\mathbb{F}_2} H^1(Q_8, \mathbb{Z}/2\mathbb{Z}) = 2, & \dim_{\mathbb{F}_2} H^2(Q_8, \mathbb{Z}/2\mathbb{Z}) = 2. \end{array}}$$

If $G = D_8$ or $Q_8$ the we expect that

$$\dim_{\mathbb{F}_2} H^1(G, \mathbb{Z}/2\mathbb{Z}) = 2$$

since ($\mathbb{Z}/2\mathbb{Z}$ has trivial $G$-action)

$$H^1(G, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}_{gr}(G, \mathbb{Z}/2\mathbb{Z})$$
$$= \mathrm{Hom}_{gr}(G/[G,G], \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

since $G/[G,G] \cong (\mathbb{Z}/2\mathbb{Z})^2$.

<u>Next time</u>: we will use spectral sequences to see that the difference between

$$\dim_{\mathbb{F}_2} H^2(D_8, \mathbb{Z}/2\mathbb{Z}) = 3, \quad \dim_{\mathbb{F}_2} H^2(Q_8, \mathbb{Z}/2\mathbb{Z}) = 2$$

can be explained by the fact that there is a <u>split</u> exact seq.

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

whereas the analogous seq.

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

does <u>not</u> split.

Alternatively, it can be explained by the fact that if we write

$$Q_8 = \langle a, b \mid a^4 = 1,\ a^2b^{-2} = 1,\ abab^{-1} = 1 \rangle$$

then the relation $a^4$ is contained in the normal closure of $\langle a^2b^{-2}, abab^{-1} \rangle$ inside the free group on $a$ and $b$, i.e., the relational structure is generated by $2 = \dim_{\mathbb{F}_2} H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ relations.